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EXPLICIT FORMULAS FOR KOROBOV POLYNOMIALS

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Abstract. In this paper we find some explicit and recurrence formulas for the Korobov polynomials and numbers. Also, author gives another explicit formulas for the degenerate Bernoulli polynomials and numbers. New recursion formulas and identities for these polynomials are also obtained.

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1. Introduction

Research in the field of obtaining explicit formulas for polynomials have received much attention by Srivastava [1, 2], Boyadzhiev [3], Cenkci [4], and Kruchinin [5, 6, 7].

The main goal of this study is to obtain several explicit formulas and identities for the Korobov polynomials and numbers.

Korobov [8] introduced special numbers P_n and polynomials $P_n(x)$ that are degenerate analogues of the Bernoulli numbers and polynomials, respectively. Ustinov [9] considered these polynomials as $K_n(x) = n!P_n(x)$ and called them the Korobov polynomials of the first kind. In [10], Ustinov rediscovered the degenerate Bernoulli polynomials (see [11]) and called them the Korobov polynomials of the second kind.

The study is conducted using the method based on generating functions and the notion of the composita.

In the papers [12, 13] authors introduced the notion of the *composita* of a given ordinary generating function $F(t) = \sum_{n>0} f(n)t^n$.

Suppose $F(t) = \sum_{n>0} f(n)t^n$ is the generating function, in which there is no free term $f(0) = 0$. From this generating function we can write the following condition

$$[F(t)]^k = \sum_{n>0} F(n,k)t^n. \tag{1}$$

The expression $F(n, k)$ is the *composita* and it is denoted by $F^{\Delta}(n, k)$.

The composita is useful for obtaining coefficients for every difficult generating function.

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2. Explicit formulas for degenerate Bernoulli polynomials and numbers

Carlitz [11, 14] defined the degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ and numbers $\beta_n(\lambda)$ by the following generating functions, respectively:

$$\frac{t(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \sum_{n\geqslant 0}\beta_n(\lambda, x)\frac{t^n}{n!}, \tag{2}$$

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \sum_{n\geqslant 0}\beta_n(\lambda)\frac{t^n}{n!}. \tag{3}$$

The first few polynomials and numbers are

$$\beta_0(\lambda, x) = 1, \beta_1(\lambda, x) = x - \frac{\lambda-1}{2}, \beta_2(\lambda, x) = x^2 - \lambda x + \frac{\lambda^2-1}{6},$$

$$\beta_3(\lambda, x) = x^3 - \frac{3(\lambda+1)}{2}x^2 + \frac{\lambda(\lambda+3)}{2}x - \frac{\lambda^2-1}{4};$$

$$\beta_0(\lambda) = 1, \qquad \beta_1(\lambda) = -\frac{\lambda-1}{2}, \qquad \beta_2(\lambda) = \frac{\lambda^2-1}{6}, \qquad \beta_3(\lambda) = -\frac{\lambda^2-1}{4}.$$

If $(1+\lambda t)^{\frac{1}{\lambda}} \to e^t$ as $\lambda \to 0$, then we get the Bernoulli polynomials

$$\frac{te^{xt}}{e^t-1} = \sum_{n\geqslant 0} B_n(x)\frac{t^n}{n!}. \tag{4}$$

Howard [15] found the explicit formula for the degenerate Bernoulli numbers

$$\beta_n(\lambda) = m!b_m\lambda^m + \sum_{j=1}^{[n/2]}\frac{n}{2j}B_{2j}s(n-1, 2j-1)\lambda^{n-2j},$$

where $m \geqslant 2$, b_n is the Bernoulli number of the second kind, B_{2j} is the Bernoulli number, and $s(n, r)$ is the Stirling number of the first kind.

He also proved some recursion formulas for the degenerate Bernoulli numbers (see Theorem 4.1; and formula 4.7 in [15]).

Reader can find other interesting properties and identities of the degenerate Bernoulli numbers related to p-adic invariant integral on Z_p in the following papers [16, 17, 18, 19].

Firstly, using a notion of the composita, we find an explicit formula for the degenerate Bernoulli numbers.

For that we represent the generating function $\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$ as the following composition of two generating functions

$$G(t) = \frac{1}{1+t} \qquad \text{and} \qquad A(t) = \left[\frac{(\lambda t+1)^{\frac{1}{\lambda}}-1}{t} - 1\right]$$

$$G(A(t)) = \frac{1}{1+\left[\frac{(\lambda t+1)^{\frac{1}{\lambda}}-1}{t} - 1\right]}.$$

Let us find a composita for the generating function $A(t)$. Since

$$[(\lambda t+1)^{\frac{1}{\lambda}}-1]^k=\sum_{n>0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\binom{\frac{j}{\lambda}}{n}\lambda^n t^n,$$

the coefficients of $\left[\frac{(\lambda t+1)^{\frac{1}{\lambda}}-1}{t}\right]^k$ are defined by

$$T(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\binom{\frac{j}{\lambda}}{n+k}\lambda^{n+k}.$$

Then the composita of the generating function $A(t)$ is

$$A^{\Delta}(n,k)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}T(n,i)$$

or

$$A^{\Delta}(n,k)=\sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{k-j}\binom{\frac{j}{\lambda}}{n+i}\lambda^{n+i}.$$

Below we present the first few terms of the composita $A^{\Delta}(n,k)$ in triangular form (the first term is $A(1,1)$):

$$\left[-\frac{p-1}{2}\right]$$

$$\left[\frac{2\,p^2-3\,p+1}{6},\frac{p^2-2\,p+1}{4}\right]$$

$$\left[-\frac{6\,p^3-11\,p^2+6\,p-1}{24},-\frac{2\,p^3-5\,p^2+4\,p-1}{6},-\frac{p^3-3\,p^2+3\,p-1}{8}\right]$$

Using the formula for coefficients of composition of generating functions from [12], we obtain an expression for the coefficients of the composition $G(A(t))=\sum_{n\geqslant 0}c_n t^n$

$$c_n=\begin{cases}1, & n=0;\\ \sum\limits_{k=1}^{n}A^{\Delta}(n,k)(-1)^k=\sum\limits_{k=1}^{n}\sum\limits_{i=0}^{k}\binom{k}{i}\sum\limits_{j=0}^{i}\binom{i}{j}(-1)^j\binom{\frac{j}{\lambda}}{n+i}\lambda^{n+i}, & n>0.\end{cases}$$

Then the explicit formula for the degenerate Bernoulli numbers is equal to

$$\beta_n(\lambda)=n!\sum_{k=1}^{n}\sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\binom{\frac{j}{\lambda}}{n+i}\lambda^{n+i}, \tag{5}$$

where $n>0$.

Therefore, we get the explicit formula for the degenerate Bernoulli polynomials

$$\beta_n(\lambda,x)=\sum_{m=0}^{n}\frac{n!}{(n-m)!}\beta_{n-m}(\lambda)\binom{\frac{x}{\lambda}}{m}\lambda^m \tag{6}$$

or

$$\beta_n(\lambda,x)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m}(\lambda)\sum_{l=0}^{m}s(m,l)x^l\lambda^{m-l}, \tag{7}$$

where $s(m,l)$ is the Stirling number of the first kind.

Next we obtain an identity that based on a compositional inverse generating function $R(t)$ of the generating function for (3).

Definition 2.1. A compositional inverse $R(t)$ of generating function $A(t)$ such that $A(t)=\sum_{n>0}a_nt^n$ and $a_1\neq 0$, is a power series such that satisfies the following condition

$$A(R(t))=t. \tag{8}$$

For obtaining a composita of $R(t)$, we use the following algorithm [13, 20]:

(1) calculate the composita $A_R^{\Delta}(n,k)$ of the reciprocal generating function of (3);
(2) calculate the composita of the compositional inverse generating function $R(t)$ by the following way

$$R^{\Delta}(n,k)=\frac{k}{n}A_R^{\Delta}(2n-k,n);$$

By the reciprocal generating function we mean the following [21]:

Definition 2.2. A reciprocal generating function $C(t)=\sum_{n\geq 0}c_nt^n$ of a generating function $B(t)=\sum_{n\geq 0}b_nt^n$, is a power series such that satisfies the following condition

$$C(t)B(t)=1. \tag{9}$$

Since the reciprocal generating function of (3) is

$$\frac{(\lambda t+1)^{\frac{1}{\lambda}}-1}{t},$$

the composita $A_R^{\Delta}(n,k)$ is equal to

$$A_R^{\Delta}(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\binom{\frac{j}{\lambda}}{n}\lambda^n.$$

Next we obtain the composita of compositional inverse generating function

$$R^{\Delta}(n,k)=\frac{k}{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\binom{\frac{j}{\lambda}}{2n-k}\lambda^{2n-k}.$$

Therefore, according to (8) and the formula for coefficients of composition of generating functions [12], we get the following identity for $n>1$

$$\sum_{k=1}^{n}\frac{\beta_{k-1}(\lambda)}{(k-1)!}\frac{k}{n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{\frac{j}{\lambda}}{2n-k}\lambda^{2n-k}=0. \tag{10}$$

If we consider the following summation

$$\frac{t}{(\lambda t+1)^{\frac{1}{\lambda}}-1}+t=\frac{t\,(\lambda t+1)^{\frac{1}{\lambda}}}{(\lambda t+1)^{\frac{1}{\lambda}}-1}$$

then we get the following formulas for the degenerate Bernoulli numbers

$$\beta_n(\lambda) = \sum_{i=0}^{n} \beta_i(\lambda) \binom{\frac{1}{\lambda}}{n-i} \lambda^{n-i}$$

or

$$\sum_{i=0}^{n-1} \beta_i(\lambda) \binom{\frac{1}{\lambda}}{n-i} \lambda^{n-i} = 0,$$

where $n > 0$.

Therefore, we obtain the recursion formula

$$\beta_n(\lambda) = -\sum_{i=0}^{n-1} \beta_i(\lambda) \binom{\frac{1}{\lambda}}{n-i+1} \lambda^{n-i+1}, \tag{11}$$

where $n > 0$.

3. Explicit formulas for Korobov polynomials and numbers

For a fixed real number p the Korobov polynomials and numbers are defined by the following generating functions, respectively [8, 10]:

$$F_K(x,t) = \frac{pt(1+t)^x}{(1+t)^p - 1} = \sum_{n\geqslant 0} K_n(p,x)\frac{t^n}{n!}, \tag{12}$$

$$F_K(t) = \frac{pt}{(1+t)^p - 1} = \sum_{n\geqslant 0} K_n(p)\frac{t^n}{n!}. \tag{13}$$

Young [22] studied the Korobov polynomials as the reciprocal polynomials of the degenerate Bernoulli polynomials and found some useful identities and recurrence relations.

The degenerate Bernoulli polynomials and the Korobov polynomials are related by the following way:

$$\beta_n(\lambda,x) = \lambda^n K_n(1/\lambda, x/\lambda), \qquad K_n(p,x) = p^n \beta_n(1/p, x/p).$$

The first few polynomials and numbers are

$$K_0(p,x) = 1, K_1(p,x) = x - \frac{p-1}{2}, K_2(p,x) = x^2 - px + \frac{p^2-1}{6},$$

$$K_3(p,x) = x^3 - \frac{3(p+1)}{2}x^2 + \frac{p(p+3)}{2}x - \frac{p^2-1}{4};$$

$$K_0(p) = 1, \quad K_1(p) = -\frac{p-1}{2}, \quad K_2(p) = \frac{p^2-1}{6}, \quad K_3(p) = -\frac{p^2-1}{4}.$$

Firstly, using a notion of the composita, we find an explicit formula for the Korobov numbers.

For that we represent the generating function $\frac{pt}{(1+t)^p-1}$ as the following composition of two generating functions

$$G(t) = \frac{1}{1+t} \qquad \text{and} \qquad A(t) = \left[\frac{(1+t)^p - 1}{tp} - 1\right]$$

$$G(A(t)) = \frac{1}{1 + \left[\frac{(1+t)^p-1}{tp} - 1\right]}.$$

Let us find a composita for the generating function $A(t)$. Since

$$[(1+t)^p-1]^k = \sum_{n>0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\binom{pj}{n}t^n.$$

the coefficients of $\left[\frac{(1+t)^p-1}{tp}\right]^k$ are defined by

$$T(n,k)=\frac{1}{p^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\binom{pj}{n+k}.$$

Then the composita of the generating function $A(t)$ is

$$A^{\Delta}(n,k)=\sum_{i=0}^{n}(-1)^{k-i}\binom{k}{i}T(n,i).$$

or

$$A^{\Delta}(n,k)=\sum_{i=0}^{n}\binom{k}{i}p^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{k+j}\binom{jp}{n+i}.$$

Below we present the first few terms of the composita $A^{\Delta}(n,k)$ in triangular form:

$$\left[\frac{p-1}{2}\right]$$

$$\left[\frac{p^2-3p+2}{6},\frac{p^2-2p+1}{4}\right]$$

$$\left[\frac{p^3-6p^2+11p-6}{24},\frac{p^3-4p^2+5p-2}{6},\frac{p^3-3p^2+3p-1}{8}\right]$$

For $k=n$ we have the following identity

$$\sum_{k=0}^{n}\binom{n}{i}p^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{n+j}\binom{jp}{n+i}=\frac{1}{2^n}(p-1)^n. \tag{14}$$

Using the formula for coefficients of composition of generating functions from [12], we obtain an expression for the coefficients of the composition $G(A(t))=\sum_{n\geqslant 0}c_nt^n$

$$c_n=\begin{cases}1, & n=0;\\ \sum\limits_{k=1}^{n}A^{\Delta}(n,k)(-1)^k=\sum\limits_{k=1}^{n}\sum\limits_{i=0}^{n}\binom{k}{i}p^{-i}\sum\limits_{j=0}^{i}\binom{i}{j}(-1)^j\binom{jp}{n+i}, & n>0.\end{cases}$$

Then the explicit formula for the Korobov numbers is equal to

$$K_n(p)=n!\sum_{k=1}^{n}\sum_{i=0}^{n}\binom{k}{i}p^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\binom{jp}{n+i}, \tag{15}$$

where $n>0$.

Therefore, we get the explicit formula for the Korobov polynomials

$$K_n(p,x)=\sum_{m=0}^{n}\frac{n!}{(n-m)!}K_{n-m}(p)\binom{x}{m} \tag{16}$$

or

$$K_n(p,x) = \sum_{m=0}^{n} \binom{n}{m} K_{n-m}(p) \sum_{l=0}^{m} s(m,l) x^l, \tag{17}$$

where $s(m,l)$ is the Stirling number of the first kind.

Next we obtain an identity that based on a compositional inverse generating function $R(t)$ of the generating function for (13).

The composita of compositional inverse generating function $R(t)$ is equal to

$$R^{\Delta}(n,k) = \frac{k}{n\,p^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{j\,p}{2\,n-k}.$$

Therefore, according to (8) and the formula for coefficients of composition of generating functions [12], we get the following identity for $n > 1$

$$\sum_{k=1}^{n} \frac{k}{p^n} \frac{K_{k-1}(p)}{(k-1)!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{j\,p}{2\,n-k} = 0. \tag{18}$$

If we consider the following summation

$$\frac{p\,t}{(t+1)^p - 1} = \frac{p\,t\ (t+1)^p}{(t+1)^p - 1} - p\,t$$

then we get the following formulas for the Korobov numbers

$$K_n(p) = \sum_{i=0}^{n} K_i(p) \binom{p}{n-i}$$

or

$$\sum_{i=0}^{n-1} K_i(p) \binom{p}{n-i} = 0,$$

where $n > 1$.

Therefore, we obtain the recursion formula

$$K_n(p) = -\frac{1}{p} \sum_{i=0}^{n-1} K_i(p) \binom{p}{n-i+1}, \tag{19}$$

where $n > 1$.

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